

FINITE GENERATION OF LIE ALGEBRAS ASSOCIATED TO ASSOCIATIVE ALGEBRAS

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ABSTRACT. Let F be a field of characteristic not 2. An associative F -algebra R gives rise to the commutator Lie algebra $R^{(-)} = (R, [a, b] = ab - ba)$. If the algebra R is equipped with an involution $*$: $R \rightarrow R$ then the space of the skew-symmetric elements $K = \{a \in R \mid a^* = -a\}$ is a Lie subalgebra of $R^{(-)}$. In this paper we find sufficient conditions for the Lie algebras $[R, R]$ and $[K, K]$ to be finitely generated.

1. INTRODUCTION

Let F be a field of characteristic not 2. An associative F -algebra R gives rise to the commutator Lie algebra $R^{(-)} = (R, [a, b] = ab - ba)$ and the Jordan algebra $R^{(+)} = (R, a \circ b = \frac{1}{2}(ab + ba))$. If the algebra R is equipped with an involution $*$: $R \rightarrow R$ then the space of skew-symmetric elements $K = \{a \in R \mid a^* = -a\}$ is a Lie subalgebra of $R^{(-)}$, the space of symmetric elements $H = \{a \in R \mid a^* = a\}$ is a Jordan subalgebra of $R^{(+)}$. Following the result of J.M. Osborn (see[4]) on finite generation of the Jordan algebras $R^{(+)}$, H , I. Herstein [4] raised the question about finite generation of Lie algebras associated to R . In this paper we find sufficient conditions for the Lie algebras $[R^{(-)}, R^{(-)}]$, $[K, K]$ to be finitely generated.

Theorem 1. *Let R be a finitely generated associative F -algebra with an idempotent e such that $ReR = R(1-e)R = R$. Then the Lie algebra $[R, R]$ is finitely generated.*

The following example shows that the idempotent condition can not be dropped.

Example 1. The algebra $R = \begin{pmatrix} F[x] & F[x] \\ 0 & F[x] \end{pmatrix}$ of triangular 2×2 matrices over the polynomial algebra $F[x]$ is finitely generated. However the Lie algebra

$$[R, R] = \begin{pmatrix} 0 & F[x] \\ 0 & 0 \end{pmatrix} \text{ is not.}$$

Key words and phrases. associative algebra, Lie subalgebra, finitely generated.

Theorem 2. *Let R be a finitely generated associative F -algebra with an involution $*$: $R \rightarrow R$. Suppose that R contains an idempotent e such that $ee^* = e^*e = 0$ and $ReR = R(1 - e - e^*)R = R$. Then the Lie algebra $[K, K]$ is finitely generated.*

The following example shows that the condition on the idempotent cannot be relaxed.

Example 2. Consider the associative commutative algebra $A = F[x, y]/id(x^2)$ with the automorphism φ of order 2: $\varphi(x) = -x$, $\varphi(y) = y$. The algebra $R = M_2(A)$ of 2×2 matrices over A has an involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} d^\varphi & b^\varphi \\ c^\varphi & a^\varphi \end{pmatrix}$. We have $[K, K] \leq xM_2(F[y])$, $\dim_F[K, K] = \infty$, which implies that algebra $[K, K]$ is not finitely generated.

W.E.Baxter [2] showed that if R is a simple F -algebra, which is not ≤ 16 dimensional over its center Z then the Lie algebra $[K, K]/[K, K] \cap Z$ is simple.

Theorem 3. *Let R be a simple finitely generated F -algebra with an involution $*$: $R \rightarrow R$. Suppose that R contains an idempotent e such that $ee^* = e^*e = 0$. Then the Lie algebra $[K, K]/[K, K] \cap Z$ is finitely generated.*

2. FINITE GENERATION OF LIE ALGEBRAS $[R, R]$

Consider the Peirce decomposition $R = eRe + eR(1-e) + (1-e)Re + (1-e)R(1-e)$. The components $eR(1-e)$, $(1-e)Re$ lie in $[R, R]$ since $eR(1-e) = [e, eR(1-e)]$, $(1-e)Re = [e, (1-e)Re]$.

Lemma 1. *The Lie algebra $[R, R]$ is generated by $eR(1-e) + (1-e)Re$.*

Proof. We only need to show that

$$[eRe, eRe] + [(1-e)R(1-e), (1-e)R(1-e)] \subseteq \text{Lie } \langle eR(1-e), (1-e)Re \rangle.$$

From $R = R(1-e)R$ it follows that $eRe = eR(1-e)Re$. Hence an arbitrary element from eRe can be represented as a sum $\sum_i a_i b_i$, $a_i \in eR(1-e)$, $b_i \in (1-e)Re$. Now

$a_i b_i = a_i \circ b_i + \frac{1}{2}[a_i, b_i]$, where $x \circ y = \frac{1}{2}(xy + yx)$. For an arbitrary element $c \in eRe$ we have $[a_i \circ b_i, c] = [a_i, b_i \circ c] + [b_i, a_i \circ c] \in [eR(1-e), (1-e)Re]$ and $[[a_i, b_i], c] = [a_i, [b_i, c]] - [b_i, [a_i, c]] \in [eR(1-e), (1-e)Re]$. We showed that

$[eRe, eRe] \subseteq [eR(1-e), (1-e)Re]$. The inclusion $[(1-e)R(1-e), (1-e)R(1-e)] \subseteq [eR(1-e), (1-e)Re]$ is proved similarly. Lemma is proved. \square

Definition 1. A pair of vector spaces (A^-, A^+) with trilinear products $A^+ \times A^- \times A^+ \rightarrow A^+, A^- \times A^+ \times A^- \rightarrow A^-, a^\sigma \times b^{-\sigma} \times c^\sigma \mapsto (a^\sigma, b^{-\sigma}, c^\sigma) \in A^\sigma$, $\sigma = +$ or $-$, is called an *associative pair* if it satisfies the identities

$$((x^\sigma, y^{-\sigma}, z^\sigma), u^{-\sigma}, v^\sigma) = (x^\sigma, (y^{-\sigma}, z^\sigma, u^{-\sigma}), v^\sigma) = (x^\sigma, y^{-\sigma}, (z^\sigma, u^{-\sigma}, v^\sigma))$$

Example 3. The pair of Peirce components $(eR(1-e), (1-e)Re)$ is an associative pair with respect to the operations $(a^\sigma, b^{-\sigma}, c^\sigma) = a^\sigma b^{-\sigma} c^\sigma$.

Lemma 2. Let R be a finitely generated algebra and let $e, f \in R$ be idempotents such that $ReR = RfR = R$. Then the associative pair $P = (eRf, fRe)$ is finitely generated.

Remark 4. In [8] it is proved that if R is a finitely generated algebra, $e \in R$ is an idempotent such that $ReR = R$ then the Peirce component eRe is a finitely generated algebra.

Proof. Suppose that the algebra R is generated by elements a_1, \dots, a_m . Suppose further that $a_i = \sum_k \alpha_{ik} u_{ik} e v_{ik} = \sum_t \beta_{it} u'_{it} f v'_{it}$, where $1 \leq i \leq m$; $\alpha_{ik}, \beta_{it} \in F$; $u_{ik}, v_{ik}, u'_{it}, v'_{it}$ are products in generators a_1, \dots, a_m . Let d denote the maximum lengths of the products $u_{ik}, v_{ik}, u'_{it}, v'_{it}$ for all i, k, t . We claim that the pair P is generated by elements $eu f, fue$, where u runs over all products in a_1, \dots, a_m of length $\leq 3d + 1$. To prove the claim we need to show that for an arbitrary product $u = a_{i_1} \cdots a_{i_N}$ of length $N > 3d + 1$ the elements $eu f, fue$ lie in the subpair generated by $ev f, fve$, where v runs over all products in a_1, \dots, a_m of length $< N$. There exist integers N_1, N_2, N_3 such that $N/3 - 1 < N_i \leq N/3, 1 \leq i \leq 3$ and

$N = N_1 + N_2 + N_3 + 2$. Let $u = u_1 a_i u_2 a_j u_3$, $\text{length}(u_i) = N_i, 1 \leq i \leq 3$. Then $a_i = \sum_k \beta_k p'_k f q'_k, a_j = \sum_t \gamma_t p''_t e q''_t$, where $\beta_k, \gamma_t \in F$; p'_k, q'_k, p''_t, q''_t are products in a_1, \dots, a_m of length $\leq d$. Now $eu f = \sum_{k,t} \beta_k \gamma_t e u_1 p'_k f q'_k u_2 p''_t e q''_t u_3 f$. The lengths of the products $u_1 p'_k, q'_k u_2 p''_t, q''_t u_3$ are less than N . The element fue is treated similarly. Lemma is proved. \square

Remark 5. We don't assume that the algebra R of Theorem 1 is unital. However passing to the unital hull we see that if R is a finitely generated algebra, $e \in R$ is an idempotent such that $ReR = R(1-e)R = R$ then the associative pair $(eR(1-e), (1-e)Re)$ is finitely generated.

We will need some definitions from Jordan theory.

Definition 2. An algebra over a field F of characteristic $\neq 2$ with multiplication $a \circ b$ is called a *Jordan algebra* if it satisfies the identities

$$\begin{aligned} \text{(J1)} \quad & x \circ y = y \circ x \\ \text{(J2)} \quad & (x^2 \circ y) \circ x = x^2 \circ (y \circ x). \end{aligned}$$

For references on Jordan algebras see [5, 7, 9].

An arbitrary associative algebra A gives rise to the Jordan algebra

$$A^{(+)} = (A, a \circ b = \frac{1}{2}(ab + ba)).$$

Definition 3. A pair of F-spaces $P = (P^-, P^+)$ with trilinear products $P^\sigma \times P^{-\sigma} \times P^\sigma \rightarrow P^\sigma$, $a^\sigma \times b^{-\sigma} c^\sigma \rightarrow \{a^\sigma, b^{-\sigma}, c^\sigma\} \in P^\sigma$; $\sigma = +$ or $-$; $a^\sigma, c^\sigma \in P^\sigma, b^{-\sigma} \in P^{-\sigma}$, is called a *Jordan pair* if it satisfies the following identities and all their linearizations:

$$\begin{aligned} \text{(J1)} \quad & \{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, z^{-\sigma}\}, x^\sigma\}, \\ \text{(J2)} \quad & \{\{x^\sigma, y^{-\sigma}, x^\sigma\}, y^{-\sigma}, z^{-\sigma}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, y^{-\sigma}\}, z^\sigma\}, \\ \text{(J3)} \quad & \{\{x^\sigma, y^{-\sigma}, x^\sigma\}, z^{-\sigma}, \{x^\sigma, y^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}. \end{aligned}$$

An arbitrary associative pair $A = (A^-, A^+)$, $a^\sigma \times b^{-\sigma} \times c^\sigma \rightarrow (a^\sigma, b^{-\sigma}, c^\sigma) \in A^\sigma$ gives rise to the Jordan pair $A^{(+)} = (A^-, A^+)$ with operations $\{a^\sigma, b^{-\sigma}, c^\sigma\} = (a^\sigma, b^{-\sigma}, c^\sigma) + (c^\sigma, b^{-\sigma}, a^\sigma) \in A^\sigma$, $\sigma = +$ or $-$.

For further properties of Jordan pairs see [6].

J.M. Osborn (see [4]) showed that a finitely generated associative algebra R gives rise to the finitely generated Jordan algebra $R^{(+)}$.

Lemma 3. *Let $A = (A^-, A^+)$ be a finitely generated associative pair. Then the Jordan pair $A^{(+)}$ is also finitely generated.*

Proof. Suppose that the pair A is generated by elements $a_1^+, \dots, a_m^+ \in A^+$; $a_1^-, \dots, a_m^- \in A^-$. Consider an (associative) product $a = a_{i_1}^+ a_{j_1}^- a_{i_2}^+ \cdots a_{j_s}^- a_{i_{s+1}}^+ \in A^+$. We claim that if the indices i_1, \dots, i_{s+1} are not all distinct then a is a Jordan expression in shorter products. We have

$$\begin{aligned} x^+ y^- u^+ v^- u^+ &= (x^+ y^- u^+ v^- u^+ + u^+ v^- x^+ y^- u^+) - u^+ v^- x^+ y^- u^+ \\ &= \{x^+ y^- u^+, v^- u^+\} - \frac{1}{2}\{u^+, v^- x^+ y^- u^+\} \end{aligned} \tag{1}$$

Linearizing this equality in u^+ (see [9]) we get

$$x^+ y^- \{u_1^-, v^-, u_2^+\} = \{x^+ y^- u_1^+ v^-, u_2^+\} + \{x^+ y^- u_2^+, v^-, u_1^+\} - \{u_1^+, v^- x^+ y^-, u_2^+\}. \tag{2}$$

Now (1) and (2) imply

$$\begin{aligned}
x^+y^-u^+v^-u^+z^-t^+ &= x^+y^-(u^+v^-u^+z^-t^+ + t^+z^-u + v^-u^+) - x^+y^-t^+z^-u^+v^-u^+ \\
&= \frac{1}{2}x^+y^-\{\{u^+v^-u^+\}, z^-, t^+\} - x^+y^-t^+z^-u^+v^-u^+ \\
&= \frac{1}{2}x^+y^-\{\{u^+v^-u^+\}, z^-, t^+\} + \frac{1}{2}\{x^+y^-t^+, z^-, \{u^+, v^-, u^+\}\} \\
&\quad - \frac{1}{2}\{\{u^+, v^-, u^+\}, z^-x^+y^-, t^+\} - \{x^+y^-t^+z^-u^+, v^-, u^+\} \\
&\quad + \frac{1}{2}\{u^+, z^-t^+y^-x^+v^-, u^+\} \tag{3}
\end{aligned}$$

If $i_p = i_q$, $p < q$, then a is an expression of the same type as the left hand sides of (1), (3). Hence a is a Jordan expression in shorter products.

We showed that the Jordan pair $A^{(+)}$ is generated by elements $a_{i_1}^+ a_{j_1}^- a_{i_2}^+ \cdots a_{i_{s+1}}^+$, where the indices i_1, \dots, i_{s+1} are distinct, and elements $a_{j_1}^- a_{i_1}^+ \cdots a_{i_s}^+ a_{j_{s+1}}^-$, where j_1, \dots, j_{s+1} are distinct. Hence the pair $A^{(+)}$ is finitely generated. \square

Proof of Theorem 1.

Proof. Let R be a finitely generated associative algebra with an idempotent e such that $R = ReR = R(1 - e)R$. By Lemma 2 the associative pair $A = (eR(1 - e), (1 - e)Re)$ is finitely generated. By Lemma 3 the Jordan pair $A^{(+)}$ is finitely generated as well. For arbitrary elements $a^\sigma, c^\sigma \in A^\sigma, b^{-\sigma} \in A^\sigma, \sigma = +$ or $-$, we have $\{a^\sigma, b^{-\sigma}, c^\sigma\} = [[a^\sigma, b^{-\sigma}], c^\sigma]$. This implies that generators of the Jordan pair $A^{(+)}$ generate the Lie algebra $A^- + [A^-, A^+] + A^+$. Now it remains to recall that $A^- + [A^-, A^+] + A^+ = [R, R]$ by Lemma 1. Theorem is proved. \square

3. FINITE GENERATION OF LIE ALGEBRAS $[K, K]$.

Let $R, * : R \rightarrow R$, be an involutive algebra with an idempotent e satisfying $ee^* = e^*e = 0$ and $ReR = R(1 - e - e^*)R = R$. Let $s = 1 - e - e^* \neq 0$. For an arbitrary element $a \in R$ we denote $\{a\} = a - a^* \in K$. Let $R_{-2} = eRe^*$, $R_{-1} = eRs + sRe^*$, $R_0 = eRe + e^*Re^* + sRs$, $R_1 = e^*Rs + sRe$, $R_2 = e^*Re$. Then $R = R_{-2} + R_{-1} + R_0 + R_1 + R_2$ is a \mathbb{Z} -grading.

Denote $K_i = K \cap R_i$, $H_i = H \cap R_i$, where $H = \{a \in R | a^* = a\}$.

Remark 6. By Lemma 2 the associative pair (R_{-2}, R_2) is finitely generated. The restriction of $*$ is an involution of the pair (R_{-2}, R_2) . However, $-*$ is also an involution of (R_{-2}, R_2) . The Jordan pair (K_{-2}, K_2) is $K((R_{-2}, R_2), *) = H((R_{-2}, R_2), -*)$.

An analog of Lemma 3 for associative pairs is not true: the Jordan pair of symmetric elements of a finitely generated involutive associate pair may be not finitely generated. An example can be derived from the Example 2 above.

Lemma 4. $K_2 = [K_1, K_1]$, $H_2 = \text{span}_F\{k^2 | k \in K_1\}$ and, similarly, $K_{-2} = [K_{-1}, K_{-1}]$, $H_{-2} = \text{span}_F\{k^2 | k \in K_{-1}\}$.

Proof. Recall that the algebra R is generated by the elements a_1, \dots, a_m . An arbitrary generator a_i can be represented as $a_i = \sum_j \alpha_{ij} v_{ij} s w_{ij}$, where $\alpha_{ij} \in F$; v_{ij} , w_{ij} are products in generators a_1, \dots, a_m (may be, empty). Let $a = a_{i_1} \cdots a_{i_r}$ be an arbitrary product of generators. Applying the equalities above to the generator a_{i_1} we get

$$\begin{aligned} e^* a e &= \sum \alpha_{i_1 j} e^* v_{i_1 j} s w_{i_1 j} a_{i_2} \cdots a_{i_r} e \\ &= \sum \alpha_{i_1 j} \{e^* v_{i_1 j} s\} \{s w_{i_1 j} a_{i_2} \cdots a_{i_r} e\} \in K_1 K_1. \end{aligned}$$

Now $K_2 = \{e^* R e\} = \{K_1 K_1\} = [K_1, K_1]$;

$$\begin{aligned} H_2 &= \{e^* a e + e^* a^* e | a \in R\} \\ &= \{k_1 k_2 + (k_1 k_2)^* | k_1, k_2 \in K_1\} \\ &= \text{span}_F\{k^2 | k \in K_1\}. \end{aligned}$$

Lemma is proved. □

Lemma 5. *There exists a finite subset $M_{-1} \subset K_{-1}$ such that $R_1 = M_{-1} R_2 + R_2 M_{-1}$. Similarly, there exists a finite subset $M_1 \subset K_1$ such that $R_{-1} = M_1 R_{-2} + R_{-2} M_1$.*

Proof. Represent each generator a_i as $a_i = \sum_j \alpha'_{ij} v'_{ij} e w'_{ij}$, where $\alpha'_{ij} \in F$; v'_{ij} , w'_{ij} are products in a_1, \dots, a_m (may be, empty). Consider an element $e^* a s$, where $a = a_{i_1} \cdots a_{i_r}$ and apply the decomposition above to a_{i_r} . We'll get

$$\begin{aligned} e^* a s &= \sum \alpha'_{i_r j} e^* a_{i_1} \cdots a_{i_{r-1}} v'_{i_r j} e w'_{i_r j} s \\ &= \sum \alpha'_{i_r j} e^* a_{i_1} \cdots a_{i_{r-1}} v'_{i_r j} \{e w'_{i_r j} s\}. \end{aligned}$$

It remains to choose $M_{-1} = \{\{e w'_{i_r j} s\}\} \subseteq K_{-1}$. Lemma is proved. □

Lemma 6. (1) $\sum_{i \neq 0} K_i \subseteq [K, K]$;
 (2) $[K, K]$ is generated by K_{-1}, K_1 .

Proof. (1) For arbitrary elements $k_1 = \{sae\}, k_{-1} = \{sae^*\}, a \in R$, we have $k_1 = [k_1, e - e^*], k_{-1} = [e - e^*, k_{-1}]$. Hence $K_{-1}, K_1 \subseteq [K, K]$. By Lemma 4, $K_2 = [K_1, K_1], K_{-2} = [K_{-1}, K_{-1}] \subseteq [K, K]$.
 (2) We have $eRe = eRe^*Re = R_{-2}R_2 = K_{-2}K_2 + K_{-2}H_2 + H_{-2}K_2 + H_{-2}H_2$. Hence $\{eRe\} \subseteq [K_{-2}, K_2] + [H_{-2}, H_2] + K_{-2} \circ H_2 + H_{-2} \circ K_2$. This implies that $\{eRe\}, K_0 \subseteq [K_{-2}, K_2] + [H_{-2}, H_2] \subseteq [K_{-2}, K_2] + [K_{-1}, K_1]$ by Lemma 4. Now sRs is spanned by elements of the type $saebs$; $a, b \in R$. We have $saebs = (sae - e^*a^*s)(ebs - sb^*e^*) - e^*a^*sb^*e^* \in K_1K_{-1} + e^*Re^*$. Hence $\{sRs\} \subseteq [K_{-1}, K_1] + \{eRe\}$. Now $\{sRs\}, K_0 \subseteq [K_{-1}, K_1] + [K_{-2}, K_2]$ by what we proved above. Lemma is proved. \square

Lemma 7. Let $n \geq 2, a_2^{(1)}, \dots, a_2^{(n+1)} \in K_2 \cup H_2; b_{-2}^{(1)}, \dots, b_{-2}^{(n)} \in K_{-2} \cup H_{-2}$ and $\text{Card}(\{b_{-2}^{(j)}, j = 1, \dots, n\}) < n$. Then

$$a_2^{(1)}b_{-2}^{(1)} \cdots a_2^{(n)}b_{-2}^{(n)}a_2^{(n+1)} \in \sum a_2^{(i_1)}b_{-2}^{(j_1)}a_2^{(i_2)} \cdots b_{-2}^{(j_r)}a_2^{(i_{r+1})}(K_{-2}K_2 + H_{-2}H_2), r < n.$$

Proof. Suppose that $a_2^{(n+1)} \in K_2$. If $b_{-2}^{(n)} \in K_{-2}$, then we are done. Let $b_{-2}^{(n)} \in H_{-2}$. Suppose that there exists $1 \leq i \leq n-1$, such that $b_{-2}^{(i)} \in K_{-2}$. Then

$$a_2^{(1)} \cdots b_{-2}^{(i)} \cdots a_2^{(n)}b_{-2}^{(n)}a_2^{(n+1)} = a_2^{(1)} \cdots \{b_{-2}^{(i)} \cdots b_{-2}^{(n)}\}a_2^{(n+1)} \pm a_2^{(1)} \cdots b_{-2}^{(n)} \cdots b_{-2}^{(i)}a_2^{(n+1)}$$

as claimed. Therefore we can assume that $b_{-2}^{(i)} \in H_{-2}, 1 \leq i \leq n$. Let $a_2^{(n)} \in H_2$. Then

$$b_{-2}^{(n-1)}a_2^{(n)}b_{-2}^{(n)}a_2^{(n+1)} = b_{-2}^{(n-1)}(a_2^{(n)}b_{-2}^{(n)}a_2^{(n+1)} - a_2^{(n+1)}b_{-2}^{(n)}a_2^{(n)}) + b_{-2}^{(n-1)}a_2^{(n+1)}b_{-2}^{(n)}a_2^{(n)}$$

which is an element of $H_{-2}H_2 + H_{-2}K_2H_{-2}H_2$. We will assume therefore that $a_2^{(n)} \in K_2$. Suppose that there exists $2 \leq i \leq n-1$, such that $a_2^{(i)} \in H_2$. Then $a_2^{(i)} \cdots a_2^{(n)}b_{-2}^{(n)}a_2^{(n+1)} = (a_2^{(i)}b_{-2}^{(i)} \cdots a_2^{(n)} + (a_2^{(i)}b_{-2}^{(i)} \cdots a_2^{(n)})^*)b_{-2}^{(n)}a_2^{(n+1)} \pm a_2^{(n)} \cdots a_2^{(i)}b_{-2}^{(n)}a_2^{(n+1)}$. Both summands on the right hand side fall into the case that has just been considered above. From now on we will assume that $a_2^{(2)}, \dots, a_2^{(n+1)} \in K_2; b_{-2}^{(1)}, \dots, b_{-2}^{(n)} \in H_{-2}$. Notice that $b_{-2}^{(i-1)}a_2^{(i)}b_{-2}^{(i)} = \{b_{-2}^{(i-1)}a_2^{(i)}b_{-2}^{(i)}\} - b_{-2}^{(i)}a_2^{(i)}b_{-2}^{(i-1)}$. The element $b_{-2} = \{b_{-2}^{(i-1)}a_2^{(i)}b_{-2}^{(i)}\}$ lies in K_{-2} . Hence the product $a_2^{(1)} \cdots a_2^{(i-1)}b_{-2}a_2^{(i+1)} \cdots a_2^{(n+1)}$ is one of those considered before. We proved that the elements $b_{-2}^{(i)}, 1 \leq i \leq n$, in $a_2^{(1)}b_{-2}^{(1)} \cdots a_2^{(n+1)}$ are skew-symmetric modulo expressions of the desired type. Now taking into account that

$\text{Card}(b_{-2}^{(i)}, 1 \leq i \leq n) < n$, we arrive at the conclusion of the lemma. We started with the assumption that $a_2^{(n+1)} \in K_2$. The case of $a_2^{(n+1)} \in H_2$ is treated similarly. This finishes the proof of the Lemma. \square

Proof of Theorem 2.

Proof. By Lemma 2 the associative pair (R_{-2}, R_2) is finitely generated. Without loss of generality we will assume that (R_{-2}, R_2) is generated by elements $a_2^{(i)}, b_{-2}^{(j)} \in K \cup H, 1 \leq i, j \leq n$. Consider the set of products $P = P_{-2} \cup P_2$, $P_2 = \{a_2^{(i_1)} b_{-2}^{(j_1)} \cdots a_2^{(i_r)} | 1 \leq r \leq n+2\}$, $P_{-2} = \{b_{-2}^{(j_1)} a_2^{(i_1)} \cdots b_{-2}^{(j_r)} | 1 \leq r \leq n+2\}$. By Lemma 4 for an arbitrary product $p \in P_{\pm 2}$ we have $p + p^* = \sum \alpha_{p,i} k_{p,i}^2$, where $\alpha_{p,i} \in F, k_{p,i} \in K_{\pm 1}$. By Lemma 5 there exist finite sets $M_{-1} \subset K_{-1}, M_1 \subset K_1$ such that $R_1 = M_{-1}R_2 + R_2M_{-1}, R_{-1} = M_1R_{-2} + R_{-2}M_1$.

We claim that the Lie algebra $[K, K]$ is generated by the union of the sets: $M_1, M_{-1}, \{[p], [q] | p, q \in P\}, \{[(p+p^*) \circ k_{q,i}, k_{q,i}] | p, q \in P\}, \{M_{-1}P_2\}, \{M_1P_{-2}\}$. By Lemma 4 and Lemma 6 it is sufficient to prove that all elements from K_{-1}, K_1 can be expressed by these elements. By Lemma 5 K_1 is spanned by elements of the type $\{k_{-1}a_2^{(i_1)}b_{-2}^{(j_1)} \cdots a_2^{(i_r)}\}, k_{-1} \in M_{-1}$. We will use induction on r .

If $r \leq n+2$ then the assumption is clear. If $r > n+2$ then by Lemma 7 applied to $a^{(i_{r-n+2})}b^{(j_{r-n+2})} \cdots a^{(i_{r-1})}b^{(j_{r-1})}a^{(i_r)}$ the element $k_{-1}a_2^{(i_1)}b_{-2}^{(j_1)} \cdots a_2^{(i_r)}$ is a linear combination of elements of the type

$k_{-1}a_2^{(\mu_1)}b_{-2}^{(\nu_1)} \cdots a_2^{(\mu_t)}k'_{-2}k'_2$ and $k_{-1}a_2^{(\mu_1)}b_{-2}^{(\nu_1)} \cdots a_2^{(\mu_t)}h'_{-2}h'_2$, where $t < r$;
 $k'_{-2} \in \{P_{-2}\}, k'_2 \in \{P_2\}; h'_{-2}, h'_2 \in \{p + p^* | p \in P\}$. We have

$$\begin{aligned} \{k_{-1}a_2^{(\mu_1)}b_{-2}^{(\nu_1)} \cdots a_2^{(\mu_t)}k'_{-2}k'_2\} &= [\{k_{-1}a_2^{(\mu_1)} \cdots a_2^{(\mu_t)}\}, [k'_{-2}, k'_2]], \\ \{k_{-1}a_2^{(\mu_1)}b_{-2}^{(\nu_1)} \cdots a_2^{(\mu_t)}h'_{-2}h'_2\} &= [\{k_{-1}a_2^{(\mu_1)} \cdots a_2^{(\mu_t)}\}, [h'_{-2}, h'_2]] \end{aligned}$$

Now it remains to notice that if $h'_2 = p + p^*, p \in P$, then $h'_2 = \sum \alpha_{p,i} k_{p,i}^2$ and

$$[h'_{-2}, k_{p,i}^2] = 2[h'_{-2} \circ k_{p,i}, k_{p,i}] \in [K_{-1}, K_1].$$

This finishes the proof of the theorem. \square

4. SIMPLE ALGEBRAS

Let R be a simple finitely generated F -algebra with an involution $*$: $R \rightarrow R$, $\text{char } F \neq 2$, e is an idemotent such that $ee^* = e^*e = 0, K = \{a \in R | a^* = -a\}$.

If $e + e^*$ is not an identity of R then the Lie algebra $[K, K]$ is finitely generated by theorem 2. Suppose that $e + e^* = 1$. As above, let $R_{-2} = eRe^*$,

$R_0 = eRe + e^*Re^*, R_2 = e^*Re$. Then $R = R_{-2} + R_0 + R_2$ is a \mathbb{Z} -grading of R .

Denote $K_i = K \cap R_i, i = -2, 0, 2$.

If R has a nonzero center Z and $\dim_Z R < \infty$ then Z is a finitely generated F -algebra. Since Z is a field it follows that $\dim_F Z < \infty$, hence $\dim_F R < \infty$ and $\dim_F [K, K] < \infty$. From now on we will assume that the algebra R is not finitely dimensional over its center.

Lemma 8. *The algebra R is generated by $K_{-2} + K_2$.*

Proof. Since the algebra R is not finite dimensional over its center it follows that R does not satisfy a polynomial identity. By the result of S. Amitsur [1] R does not satisfy a polynomial identity with involution. Hence $[[[K, K], K], [[K, K], K]] \neq (0)$. I. Herstein [4] proved that if A is a subalgebra of R such that $[A, K] \subseteq A$, A is not commutative and $\dim_Z R > 16$, then $A = R$. Applying this result to the associative subalgebra generated by $[[K, K], K]$ we see that $[[K, K], K]$ generates R .

Let us show that $[[K, K], K] \subseteq K_{-2} + [K_{-2}, K_2] + K_2$. The Lie algebra $K_{-2} + [K_{-2}, K_2] + K_2$ is an ideal in the Lie algebra $K = K_{-2} + K_0 + K_2$ and, hence, in the Lie algebra $[K, K]$. As shown by W. E. Baxter [2] the Lie algebra $[K, K]/[K, K] \bigcap Z$ is simple. Hence $[K, K] \subseteq K_{-2} + [K_{-2}, K_2] + K_2 + Z$. Now, $[[K, K], K] \subseteq [K_{-2} + [K_{-2}, K_2] + K_2, K] \subseteq K_{-2} + [K_{-2}, K_2] + K_2$. This finished the proof of the lemma. \square

Lemma 9. *Let A be a semi prime F -algebra with an involution $*$: $A \rightarrow A$, $\text{char } F \neq 2$, $K = K(A, *) = \{a \in A \mid a^* = -a\}$. Suppose that $k \in K$ and $kKk = (0)$. Then $k = 0$.*

Proof. Let $H = \{a \in A \mid a^* = a\}$. Clearly, $A = K + H$. If $k \neq 0$ then $kAk = kHk \neq (0)$. Choose an element $h \in H$ such that $khk \neq 0$. We have $(khk)K(khk) = (0)$. For an arbitrary element $h_1 \in H$ we have $khkh_1khk = k(hkh_1 + h_1kh)khk - kh_1khkhk = 0$, since $hkh_1 + h_1kh \in K$ and $hkh \in K$. This implies $(khk)A(khk) = (0)$, which contradicts the semi primeness of A . Lemma is proved.

Let $F \langle X \rangle = F \langle x_1, \dots, x_m \rangle$ be the free associative algebra without 1. The mapping $x_i \rightarrow -x_i, 1 \leq i \leq m$, extends to the involution $*$: $F \langle X \rangle \rightarrow F \langle X \rangle$. For an arbitrary generator x_i , arbitrary elements $a_1, a_2, a_3 \in K(F \langle X \rangle, *)$ denote $f(x_i, a_1) = x_i a_1 x_i, f(x_i, a_1, a_2) = f(x_i a_1) a_2 f(x_i, a_1), f(x_i, a_1, a_2, a_3) = f(x_i, a_1, a_2) a_3 f(x_i, a_1, a_2)$.

Let I be the ideal of the algebra $F \langle X \rangle$ generated by all elements $f(x_i, a_1, a_2, a_3), 1 \leq i \leq m, a_1, a_2, a_3 \in K(F \langle X \rangle, *)$.

Recall that the Baer radical $B(A)$ of an associative algebra A is the smallest ideal of A such that the factor-algebra $A/B(A)$ is semi prime. The Baer radical is locally nilpotent: an arbitrary finite collection of elements from $B(A)$ generates a nilpotent subalgebra (see [3]). \square

Lemma 10. *The factor-algebra $F < X > / I$ is nilpotent.*

Proof. Since $I^* = I$, the involution $*$ gives rise to an involution on $A = F < X > / I$. Let $B(A)$ be the Baer radical of A , $\overline{A} = A/B(A)$. The radical $B(A)$ is invariant with respect to any involution, hence \overline{A} is an involutive semi prime algebra. By Lemma 9 the image of an arbitrary element $f(x_i, a_1, a_2, a_3)$, $a_1, a_2, a_3 \in K(F < X >, *)$, is equal to zero in \overline{A} . Again, subsequently applying lemma 9 three times we get $f(x_i, a_1, a_2) = 0$ in \overline{A} , $f(x_i, a_1) = 0$ in A and, finally, $x_i = 0$ in \overline{A} , which means that $A = B(A)$. Since the algebra A is finitely generated we conclude that A is nilpotent. Lemma is proved. \square

The degree of nilpotency of the algebra A depends on the number of generators m . Let $F < x_1, \dots, x_m >^{d(m)} \subseteq I$.

Now let us return to our finitely generated simple algebra R . By Lemma 8 there exists elements $k_1^+, \dots, k_m^+ \in K_2$, $k_1^-, \dots, k_m^- \in K_{-2}$ that generates R .

Lemma 11. *The Jordan pair (K_{-2}, K_2) is generated by elements $\{a\}$, where a are products in k_i^\pm , $1 \leq i \leq m$ generated by products of odd length $< d(2m)$.*

Proof. Choose a generator k_i^σ , $\sigma = +$ or $-$, and three elements $c_1, c_2, c_3 \in K_{-\sigma 2}$. Denote $f_1 = k_i^\sigma c_1 k_i^\sigma$, $f_2 = f_1 c_2 f_1$, $f_3 = f_2 c_3 f_2$.

Choose arbitrary elements $a_1, \dots, a_n \in K_{-\sigma 2}$; $b_1, \dots, b_n \in K_{\sigma 2}$.

- (1) Let $u = a_1 b_1 \dots a_n b_n$. Then $f_1 u = k_i^\sigma c_1 k_i^\sigma u = k_i^\sigma c_1 \{k_i^\sigma u\} \pm k_i^\sigma (u^* c_1) k_i^\sigma$. Hence $\{f_1 u\} = \{k_i^\sigma, c_1, \{k_i^\sigma u\}\} \pm \{k_i^\sigma, \{c_1 u\}, k_i^\sigma\}$ is a nontrivial Jordan expression.
- (2) Let $u = b_1 a_1 \dots b_t a_t$, $v = a_{t+1} b_{t+1} \dots a_n b_n$. Then $\{u f_2 v\} = \{\{u f_1\}, c_2, \{f_1 u\}\} \pm \{f_1 \dots\}$ is a nontrivial Jordan expression by (1).
- (3) Let $u = a_1 b_1 \dots b_{t-1} a_t$, $v = a_{t+1} b_{t+1} \dots a_{n-1} b_{n-1} a_n$, it remains to consider the product $u f_3 v$. We have $f_3 = f_2 c_3 f_2 = f_1 c_2 f_1 c_3 f_1 c_2 f_1 = k_i^\sigma c_1 k_i^\sigma c_2 k_i^\sigma c_1 k_i^\sigma c_3 k_i^\sigma c_1 k_i^\sigma c_2 k_i^\sigma c_1 k_i^\sigma = k_i^\sigma c_2' k_i^\sigma c_3 k_i^\sigma c_2' k_i^\sigma$, where $c_2' = c_1 k_i^\sigma c_2 k_i^\sigma c_1$.

$$\text{Hence, } \{u k_i^\sigma c_2' k_i^\sigma c_3 k_i^\sigma c_2' k_i^\sigma v\} = \{\{u k_i^\sigma c_2'\}, k_i^\sigma c_3 k_i^\sigma, \{c_2' k_i^\sigma v\}\} \pm \{c_2' \dots\}$$

The second summand can be treated in the same way as we did in (1).

Now we are ready to finish the proof of the lemma

Let $a = k_{i1}^\sigma, k_{j1}^{-\sigma} \dots k_{is}^\sigma$, $2s-1 \geq d(2m)$. Then by lemma 11 a is a Jordan expression in elements b , where b are products in k_i^\pm , $1 \leq i \leq m$, of odd length less than $2s-1$. This proves the lemma. \square

As we have already mentioned above $[K, K] + Z/Z = K_{-2} + [K_{-2}, K_2] + K_2 + Z/Z$. In view of Lemma 11 this implies that the algebra $[K, K]/[K, K] \bigcap Z$ is finitely generated. Theorem 3 is proved.

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